

Student seminar exercise sheet Week 5

1. Let A be an open subset of \mathbb{C} and let (f_n) be a sequence of holomorphic functions on A that converges uniformly on every compact subset to a function f .

Show that f is holomorphic on A and that the sequence of derivatives (f'_n) converges uniformly on all compact subsets to f' .

Hint: Let D be a closed disk contained in A and let C be its boundary. For s_0 in the interior of D , Cauchy's Formula applies to the $f_n(s_0)$. Let $n \rightarrow \infty$ to get a similar formula for $f(s_0)$. For the derivatives, proceed in the same way, using

$$f'(s_0) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - s_0)^2} ds.$$

2. Prove the following Lemma:

Lemma 0.1. *If $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $s = s_0$, then it converges uniformly in every domain of the form*

$$\{s : \operatorname{Re}(s - s_0) \geq 0, |\operatorname{Arg}(s - s_0)| \leq \theta\}$$

with $\theta < \frac{\pi}{2}$.

Hint: Assume $s_0 = 0$. Show that we can replace domains of the form

$$\{s : \operatorname{Re}(s) \geq 0, |\operatorname{Arg}(s)| \leq \theta\}$$

for $\theta < \frac{\pi}{2}$ by domains of the form

$$\{s : \operatorname{Re}(s) \geq 0, \frac{|s|}{\operatorname{Re}(s)} \leq M\}$$

and prove the result on it. To prove the result, start by using Abel's Lemma and then use the following fact to obtain an upper bound to $|S_{m,r}|$ in Abel's Lemma:

$$|e^{-cs} - e^{-ds}| = |s| \int_c^d e^{-t\operatorname{Re}(s)} dt = \frac{|s|}{\operatorname{Re}(s)} (e^{-c\operatorname{Re}(s)} - e^{-d\operatorname{Re}(s)}).$$

3. Show that

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{\gamma_n}{n^s}, \quad \gamma_n = \#\{\mathfrak{a} : N\mathfrak{a} = n\}$$

is absolutely convergent for $\operatorname{Re}(s) > 1$. (Compare this to (i) of Corollary 3.3 in the section on Dirichlet series—are the γ_n bounded?)

4. Let $p, m \in \mathbb{N}$ with p prime such that $p \nmid m$. Our goal is to prove that

$$(1 - T^f)^{\varphi(m)/f} = \prod_{\chi \in (\mathbb{Z}/m\mathbb{Z})^\times} (1 - \chi(p)T) \quad \forall T \in \mathbb{C}$$

where $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$ and f is the order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$.

(a) Let $K = \mathbb{Q}(\zeta_m)$, $G = \operatorname{Gal}(K/\mathbb{Q})$, and $Z = Z(p)$ the decomposition group of p . Show that the restriction map $\widehat{G} \rightarrow \widehat{Z}$ is surjective.

(b) Deduce that

$$\prod_{\chi \in \widehat{G}} (1 - \chi(p)T) = \left(\prod_{\psi \in \widehat{Z}} (1 - \psi(p)T) \right)^g,$$

where $g = \#\ker(\widehat{G} \rightarrow \widehat{Z}) = \varphi(m)/f$.

(c) Identify $\widehat{Z} \cong \mu_f$ via $\psi \mapsto \psi(p)$ and use

$$\prod_{\eta \in \mu_f} (1 - \eta T) = 1 - T^f$$

to conclude.